Adaptive Free Boundary Solutions to Non- Newtonian Fluid Boundary Layer Flows on a Flat Moving Wall

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Abstract: *The boundary-layer flow of a non-Newtonian fluid over a flat moving surface is simulated, by solving the governing equations, using an efficient implicit finite difference scheme called the Keller's box scheme. The effect of non-Newtonian viscosity of the fluid is taken into account by incorporating appropriate viscosity models. The effect of the moving surface velocity, affects the wall shear stress in a complex manner, inducing flow separation in certain cases. One of the challenges in obtaining numerical solutions to the present problem is the representation of the extent of the 'infinite' domain on the computer. In this paper, the method suggested by Fazio, is used to tackle this issue, in which, the location of the upper boundary is treated as an additional variable and this is obtained as a part of the solution. The results obtained by such an approach, agree well with results available using other methods, such as iterative transformation method used by Bognar. The extent of the upper boundary, which is obtained as part of solution, is slightly different and it seems to be a strong function of the power law index, but a weak function of the wall velocity. The results, in terms of skin friction coefficient, compare well with other published data and these are very useful in assessing the behaviour of different non-Newtonian fluids for practical engineering applications.*

Keywords: **Non Newtonian fluids, boundary layer flows, Keller's box scheme, numerical analysis**

1. INTRODUCTION

Non-Newtonian fluids, by definition, refer to a class of fluids that do *not* exhibit a linear relationship between the shear stress and the strain rate, as postulated by the Newtonian law of viscosity. Such fluids, like foams, emulsions, pulps, slurries and polymeric melts, are of very great significance and practical relevance in various industrial and manufacturing processes. The spectrum of behaviour of various non-Newtonian fluids is so vast that Newtonian behaviour always seems to be an exception rather than a rule. During the recent times, the study of non-Newtonian fluids has gained vigorous all-round interest and attention, because of its numerous technological applications such as production of plastic sheets, performance of different kinds of lubricants and study of flow of biological fluids, like blood flow in veins. A basic introduction to non-Newtonian fluids can be found in [1]. In particular, the study of flow of an incompressible non-Newtonian fluid over a moving surface has attracted several studies due to its relevance in various specific applications, like extrusion of polymer sheets from a dye or in the production of plastic films.

In the present paper, the two dimensional boundary-layer equations for laminar incompressible flow over a moving flat plate are formulated. The plate surface is assumed to be moving with a velocity U_w. Similarity transformations are used to transform the governing partial differential equations of conservation of mass and momentum into a system of ordinary differential equations. The resulting third order system of equations are reformulated as a system of three first order equations and solved numerically by an accurate and efficient implicit finite difference scheme. By using the adaptive free boundary technique proposed by Fazio [2], the extent of the boundary layer is also got as a part of the solution.

2. Governing Equations and Methodology

The two dimensional boundary layer equations can be derived, by applying the boundary layer approximations, to the full Navier-Stokes equations.

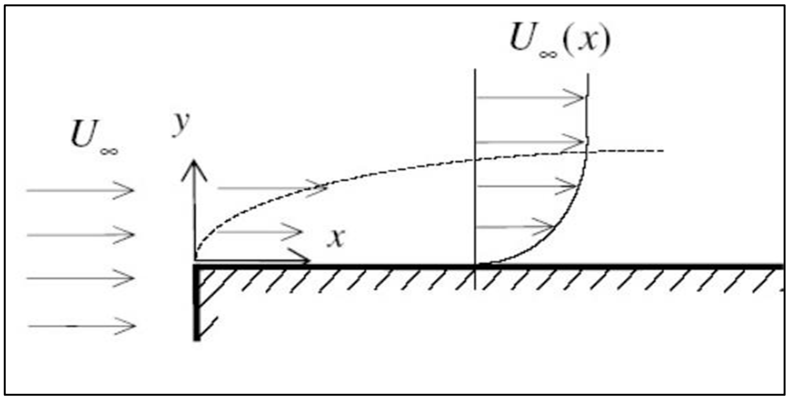

Figure 1. Two Dimensional Boundary Layer Flow over a Flat Surface

Referring to Fig.1, the momentum equation in x-direction can be written as

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}\right) \quad (1)$$

and the y-momentum equation simply implies that the pressure gradient in the normal direction $\partial p/\partial y$ is zero. In addition, we have the continuity equation for an incompressible fluid as given below

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (2)$$

Here u and v are the velocities in the x and y directions respectively. The boundary conditions for solving the above equations are the prescribed wall velocity U_w at the wall and velocity matching with the external invisicid flow velocity U_∞ at the edge of the boundary layer. Mathematically, these translate into the following equations.

$u = U_w$ and $v = 0$ at y = 0 (at the wall)

$u = U_\infty$ as y tends to ∞ (at the boundary layer edge) (3)

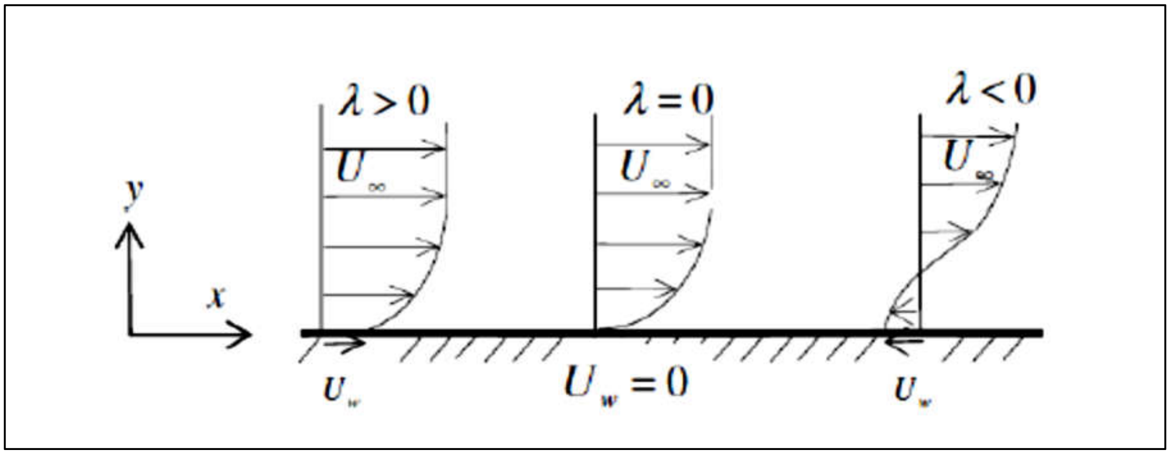

Figure 2. Velocity Profiles in the Boundary Layer for Different Wall Velocities

When U_W is negative, the plate is moving in the opposite direction to the flow over the plate. The velocity profiles in the boundary layer for different values of λ (where $\lambda = U_w / U_\infty$) are shown in Fig. 2

3. Power Law Viscosity Model for Non-Newtonian Fluids

For Newtonian fluids, the shear stress τ_{xy} (= τ_{yx}) is proportional to the strain rate γ (= ∂u/∂y) and the proportionality constant is the coefficient of viscosity η as given by

$$\tau_{xy} = \eta \, \gamma \tag{4}$$

For non-Newtonian fluids, the coefficient of viscosity itself is a function of the strain rate and hence

$$\tau_{xy} = \eta(\gamma)\, \gamma \tag{5}$$

which leads to a non-linear relationship between shear stress and the strain rate. Various viscosity models exist in the literature and the simplest model is the power law model in which η(γ) is taken as

$$\eta(\gamma) = k\,(\gamma)^{(n-1)} \tag{6}$$

k is called the consistency coefficient and n is the power-law index. Fluids, for which n > 1, such as corn syrup, and quick sand are called *dilatant fluids* or shear thickening fluids and those for which n < 1, such as paints and silicone oils, are called as *pseudo-plastics* or shear thinning fluids. n=1, of course, represents the ideal Newtonian fluid. For the power-law fluids, τ_{xy} can be written in terms η and γ, as given by equations (5) and (6) as

$$\tau_{yx} = k \left|\frac{\partial u}{\partial y}\right|^{n-1} \frac{\partial u}{\partial y} \tag{7}$$

and hence equation (1) can be written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu_c \left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right) \quad (8)$$

where $\mu_c = k/\rho$. The continuity equation will be satisfied automatically if we introduce a stream function ψ and consequently, the velocity components, in terms of 'ψ' are given by

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \quad (9)$$

The x-momentum equation is transformed, using the similarity transformation, in which, the variables x and y are combined into a single non-dimensional similarity variable 'η' defined as

$$\eta = Re^{(1/n+1)}(y/x), \text{where } Re = U_\infty^{(2-n)}\, x/(\mu_e) \quad (10)$$

This leads to a non–dimensional stream function ' f ', which is related to ψ as follows,

$$\psi(x,y) = \mu_c^{\frac{1}{n+1}}(U_\infty)^{\frac{2n-1}{n+1}}x^{\frac{1}{n+1}}f(\eta) \quad (11)$$

Using the above, the momentum equation (8), is now transformed into a single third order ordinary differential equation in ' f ' as follows

$$(|f''|^{n-1}f'')' + \frac{1}{n+1}ff'' = 0 \quad (12)$$

The boundary conditions as given in equation (3), now reduce to

$$f(0) = 0 \text{ and } f'(0) = -\lambda \text{ and } f'(\infty) = 1 \quad (13)$$

It should be noted that if λ is positive then the plate and the free stream move in opposite directions and if λ is negative then the plate and free stream move in same direction. Equation (12) can be considered as generalised Blasius equation applicable for power law fluids. The similarity variable η can also be written in terms of a Reynolds number Re_x defined as

$$Re_x = \rho U_\infty{}^{2-n}x^n\mu_c{}^{-1} \quad (14)$$

and where the non-dimensional ' η ' coordinate is given by

$$\eta = Re_x^{\frac{1}{n+1}} \quad (15)$$

With the above formulation, the velocity components u and v can be written in terms of f as shown below

$$u(x,y) = U_\infty f'(\eta)$$

$$v(x,y) = \frac{U_\infty}{n+1}Re_x^{(n+1)}\left(\eta f'(\eta) - f(\eta)\right) \quad (16)$$

Thus f '(η) represents the non-dimensional velocity u/U_∞. The quantity f "(0) has a physical significance. It represents the skin friction drag force or the fluid dynamic resistive force, which is created due to the flow on the surface of the body. If we define f ”(0) = γ as the skin friction parameter, then the non-dimensional skin friction drag coefficient C_D can be written in terms of γ as follows

$$C_D = (n+1)^{\frac{1}{n+1}} \mathrm{Re}^{\frac{-n}{n+1}} |\gamma|^{(n-1)} \gamma \tag{17}$$

The wall shear stress τ_w (x) can be written as

$$\tau_w(x) = \left[\frac{\rho^n K U_\infty{}^{3n}}{x^n}\right]^{\frac{1}{n+1}} |\gamma|^{n-1} \gamma \tag{18}$$

Equation (12) along with the boundary conditions prescribed by equation (13) constitutes a typical non-linear two point boundary value (NTPBV) problem defined in the semi-infinite region (0, ∞). Beam deflection problems in structural engineering and problems that occur in modelling chemical reactions, are also of similar type. Analytical solutions to such problems are difficult to obtain, even though the use Homotopy Perturbation Method has been recently used for simpler problems [3]. Numerical techniques such as shooting methods can be used, in which, the boundary value problem is converted into an initial value problem by guessing the values of the unknown variable at one boundary and marching to other infinite boundary using Rung-Kutta integration. The unknown boundary values are iterated progressively, till the values specified, at the infinite boundary are satisfied within a prescribed tolerance. This method becomes cumbersome and practically impossible to apply, when more than one or two boundary values are to be initially guessed and the correct values which satisfy the boundary conditions at the infinite boundary are to be finally got by trial and error. Hence one has to use more efficient techniques, such as finite difference methods, to get accurate solutions to more difficult problems.

4. Solving NTPBV Problems in Semi-Infinite Domains

One of the challenges faced, while solving NTPBV problems numerically in infinite or semi-infinite regions, on the computer, is the placement of ‘*infinite’* boundary, at a finite value. Usually the boundary is placed arbitrarily at some *acceptable and widely used value*. By trial and error, the final value is fixed, by satisfying the boundary conditions that are to be imposed at these boundaries. In this paper, the method proposed by Fazio [2] is used, in which the value of the extent of unknown boundary, is treated as an additional variable and the computational domain is transformed into a standard region extending from 0 to 1. The governing n^{th} order ordinary differential equation is formulated as a system of ‘n‘ first order equations. The first order derivatives occurring in these equations, are replaced by a highly accurate finite difference representation, called Keller’s box scheme [4] and the resulting system of algebraic equations, which are in block tri-diagonal form, are solved by the efficient Thomas algorithm. The conventional two-dimensional boundary layer equations are transformed by using similarity transformation into a higher order ordinary differential equation. This equation is a third order non linear ordinary differential equation. Keller box scheme is second order accurate even on grids with non uniform spacing. Hence it is most suitable for boundary layer flow problems. In these problems, the gradients near the wall are very large compared to the values near the edge of the boundary layer. These gradients have to be resolved accurately by using small grid spacing near the wall and progressively larger spacing away from the wall as shown in Fig.3.

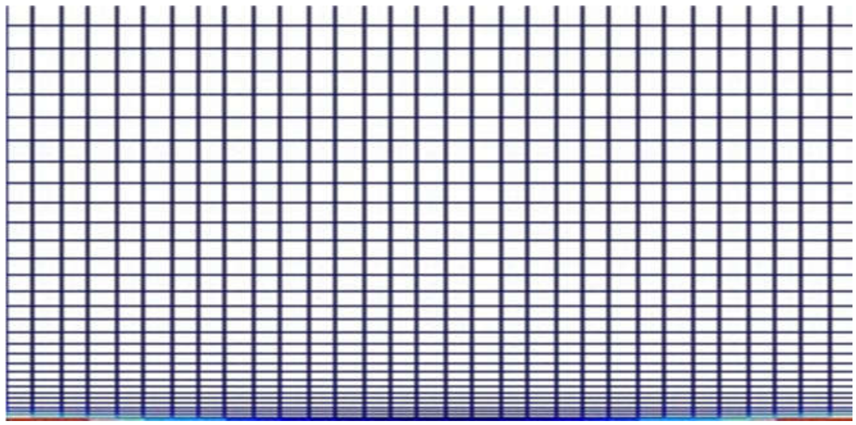

Figure 3. Non-uniform Grid Spacing in the Boundary Layer

4.1 Keller's Box Scheme

Keller box method is a very efficient implicit finite difference scheme. This method has already been successfully applied to several nonlinear problems governed by parabolic partial differential equations. It is much faster, easier to program and is most flexible of all the other common methods, being easily adaptable to solving equations of any order. In Keller box method, the system of governing nonlinear ordinary differential equations is written in the form of a system of first order equations. The first order derivatives with respect to "η" occurring in these equations are replaced by a second order accurate centred difference formula. which are taken about the centre of each grid cell. Thus the method is applicable to grids with non-uniform spacing and have a truncation error of the order of $\Delta h^2/24$.This scheme is implicit with second order accuracy in both space and time and allows the step size of time and space to be arbitrary (non-uniform). The disadvantage of the method is that the computational effort per time step is rather high due to the replacement of the higher order derivative by a set of first derivatives. For example in the present case equation (12) is formulated as a system of three first order equations

$$f' = u$$

$$u' = v$$

$$v' = \frac{-f(f^{"})^{2-n}}{(n+1)} - (n-1) \qquad (19)$$

Each of the above equations can be written in finite difference form as explained above. The domain η (0 , ∞) is divided into a set of non-uniformly spaced grids by specifying the first grid spacing and multiplying each successive grid spacing by a factor K which is chosen as greater than 1 If Δz_1 is the first grid spacing then $\Delta\eta_2 = k\,\Delta\eta_1$. Generalising this we have

$$\Delta\eta_j = k\Delta\eta_{j-1} \qquad (20)$$

The j^{th} grid point η_j can be written in terms of first step size $\Delta\eta_1$

$$\eta_j = \Delta\eta_1\left[\frac{k^{(j-1)}-1}{k-1}\right] \tag{21}$$

The boundary conditions given by equation (14) now read as

$$f(0) = 0\ ,\ u(0) = -\lambda \text{ and } u(\infty) = 0 \tag{22}$$

4.2 Implementation of Adaptive Grid

In the adaptive grid method, the computational domain is taken as varying from 0 to 1 by defining a new variable Z as

$$Z = \eta / \eta_\infty \tag{23}$$

By using chain rule,

$$\frac{df}{dZ} = \frac{df}{d\eta}\frac{d\eta}{dZ} = \eta_\infty\frac{df}{d\eta} \tag{24}$$

With similar expression for u and v. the three equations for u , v and v' are written as,

$$u = \eta_\infty f'$$

$$v = u' = \eta_\infty f''$$

$$v' = u'' = f''' = \eta_\infty\left[\frac{-f(f'')^{2-n}}{(n+1)} - (n-1)\right] \tag{25}$$

An additional equation for η_∞ is added to the above system of equations.

$$\frac{d\eta_\infty}{dZ} = 0 \tag{26}$$

as η_∞ is constant and does not depend on z. The computational domain η which varies from 0 to η_∞ is now mapped to the region between z = 0 (plate surface) to z = 1 (boundary layer edge). The boundary conditions given by equation (23) now transform as

$$f(0) = 0\ \ u(0) = 0\ \ u(1) = 1 \tag{27}$$

As the system order has become 4, an additional boundary condition is needed to solve the system of equations .This is introduced by the requirement that the shear stress at boundary layer edge is zero i.e. the shear stress coefficient f ''(1) = v(1) = 0. In practice this is implemented as

$$f''(1) = v(1) \rightarrow \delta \quad \text{where } \delta \approx 10^{-4} \text{ or lower} \tag{28}$$

This condition is added to the other three boundary conditions given by equation (27). In practical calculations, since the outer boundary is fixed as Z = 1, we normally select the number grid points J, the multiplication factor k and determine the first grid spacing $\Delta\eta_1$ from equation (22) by putting j = J.

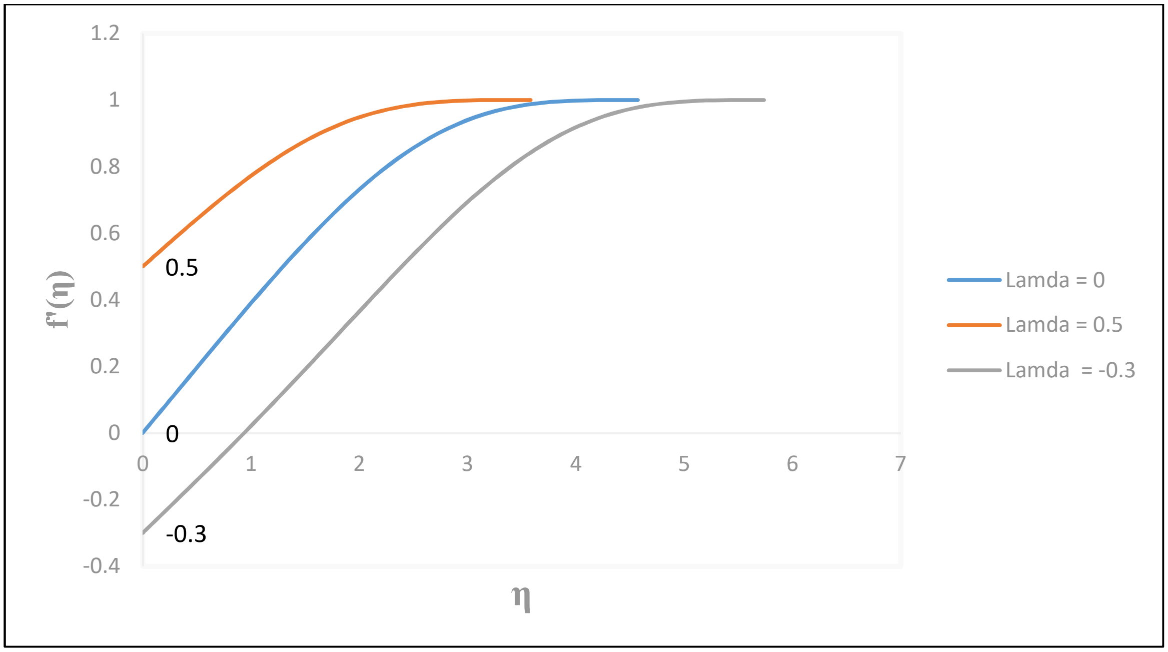

Figure 4. Velocity profiles for different λ values for n=1.3

5. Results and Discussion

Data has been generated for the velocity profiles and shear stress profiles in the boundary layer for various values of power law index 'n' and the wall velocity ratio 'λ'. The extent of the boundary layer, as indicated by the value of η_{∞}, is also computed as part of the solution. The range of ' n ' in the present study is taken as 0.5 to 2.0 and the range of ' λ ' is taken as varying from - 0.3 to 0.7. It is found that there is a critical value of ' λ ' (= λ_c) beyond which there is flow separation and the boundary layer assumptions break down. It is also observed that there are dual solutions for certain range of ' λ ' values. Figures 4 and 5 present the velocity profiles for typical dilatants and pseudo plastic fluid with n = 1.3 and 0.8 respectively, for different wall velocity ratios ' λ ' . It is seen that the boundary layer thickness is much more for pseudo plastic fluids than dilatants fluids. Figures 6, 7 and 8 present the shear stress profiles for different values of wall velocity ratios ' λ ' for typical dilatants, Newtonian and pseudo plastic fluids with n = 1.3, 1.0 and 0.8 respectively. for different wall velocity ratios ' λ ' . It is seen that wall shear stress f "(0) is more pseudo plastics than dilatants. For Newtonian fluids, the wall shear stress obtained by the present method exactly matches with the Blasius value of 0.33206. Figures 9 and 10 show the effect of viscosity index ' n ' on the shear stress profiles for two values of λ = - 0.5 and 0.3. It is seen that wall shear stress decreases as ' n ' increases. These results are also consistent with values from reference literature.

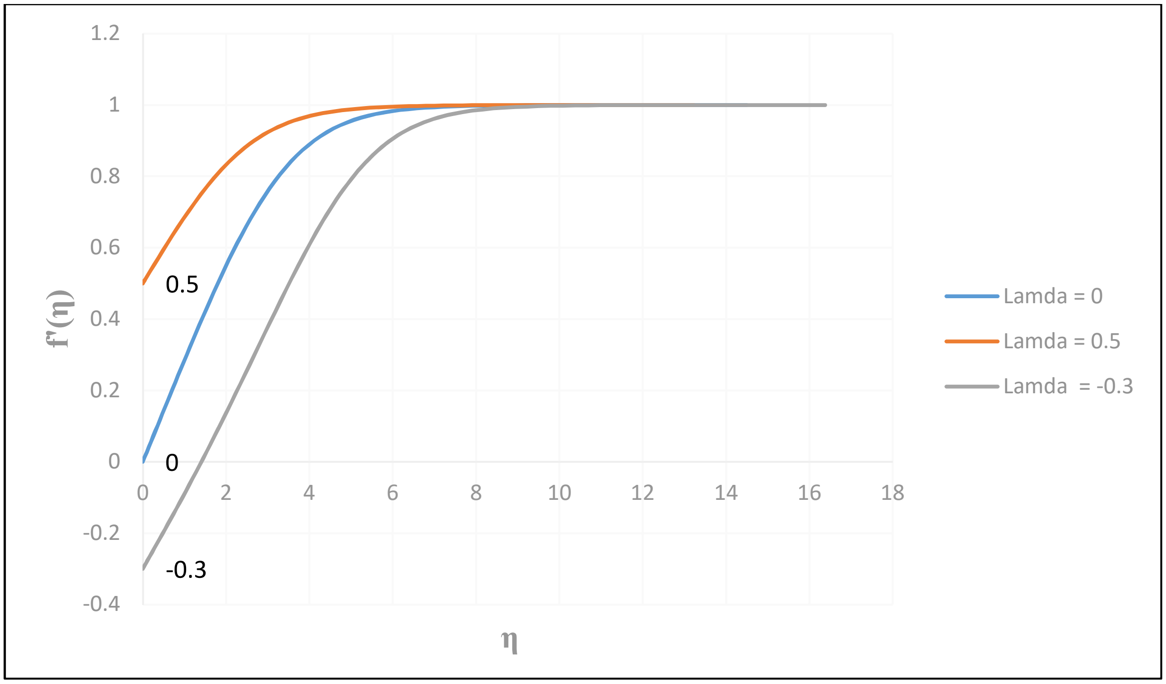

Figure 5. Velocity profiles for different λ values for n=0.8

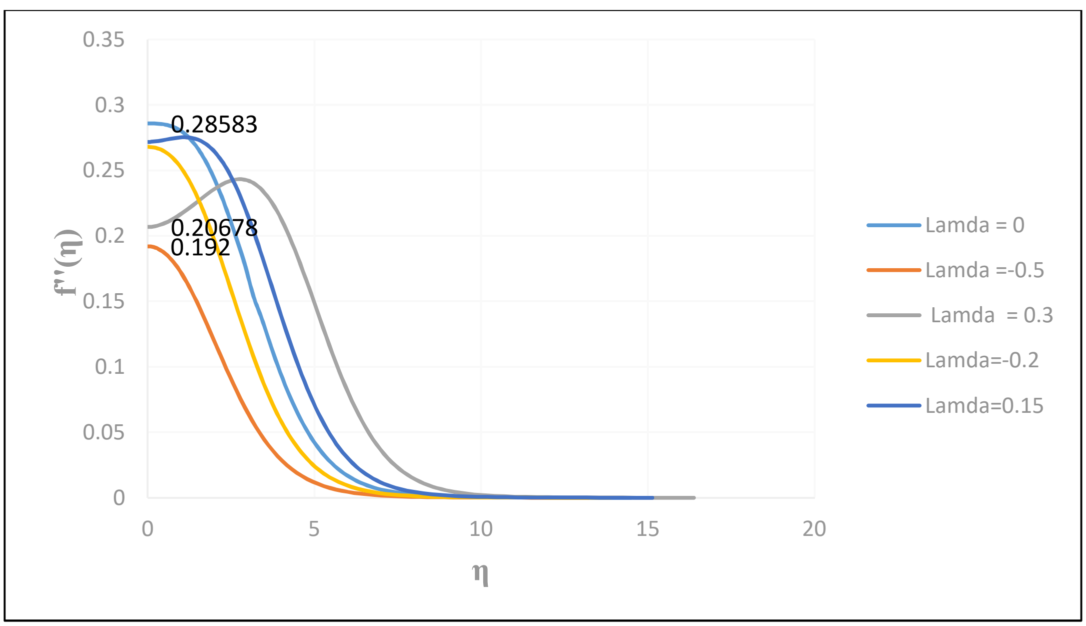

Figure 6. Shear stress profiled for different λ values for n=1.3

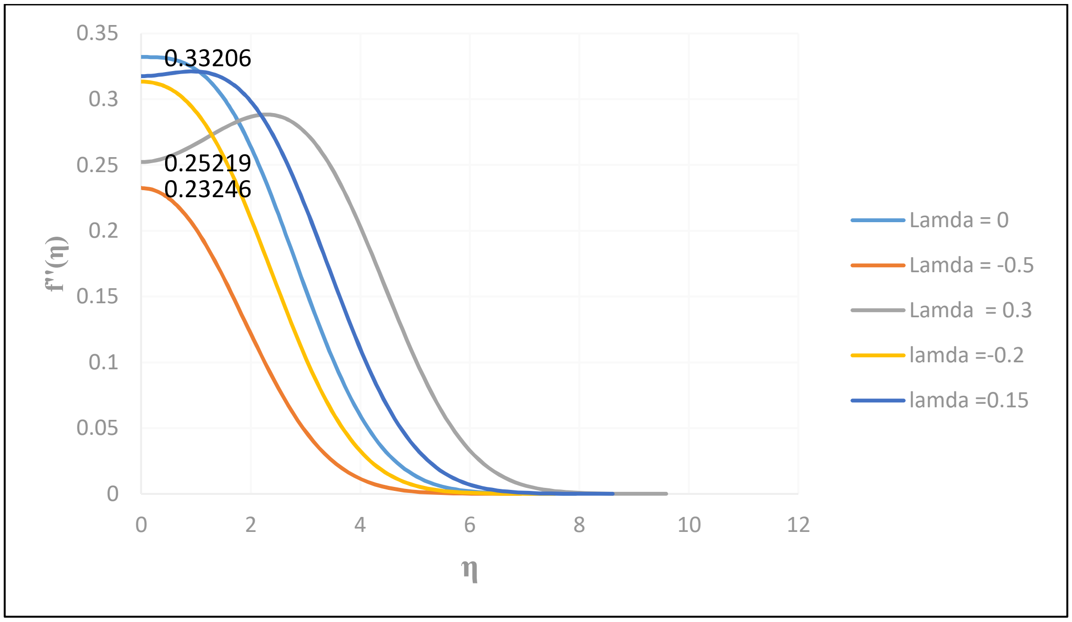

Figure 7. Shear stress profiled for different λ values for n=1.0

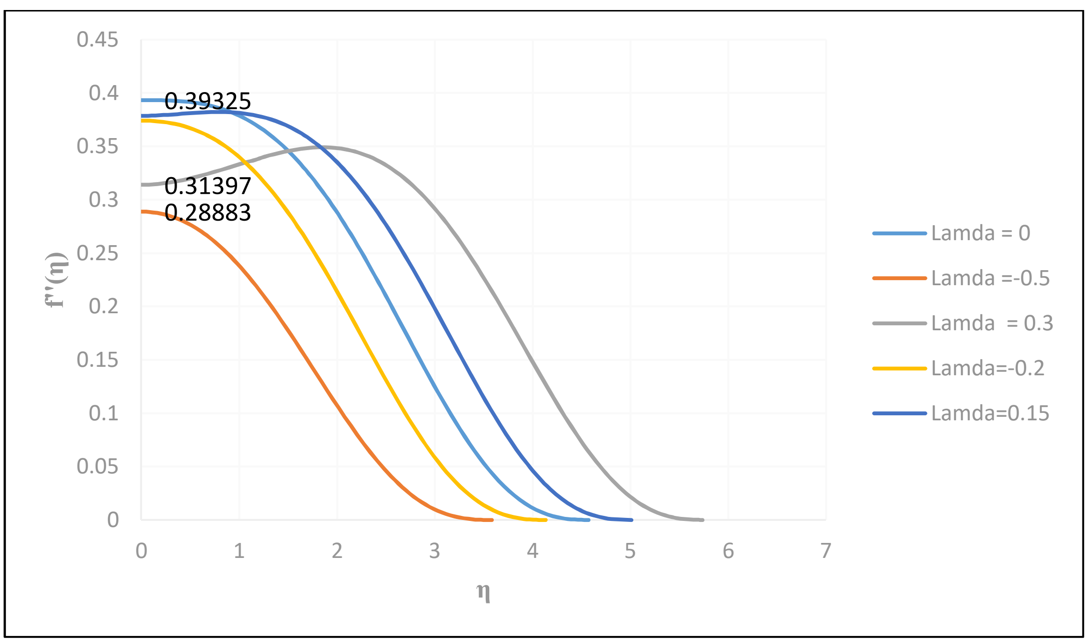

Figure 8. Shear stress profiled for different λ values for n=0.8

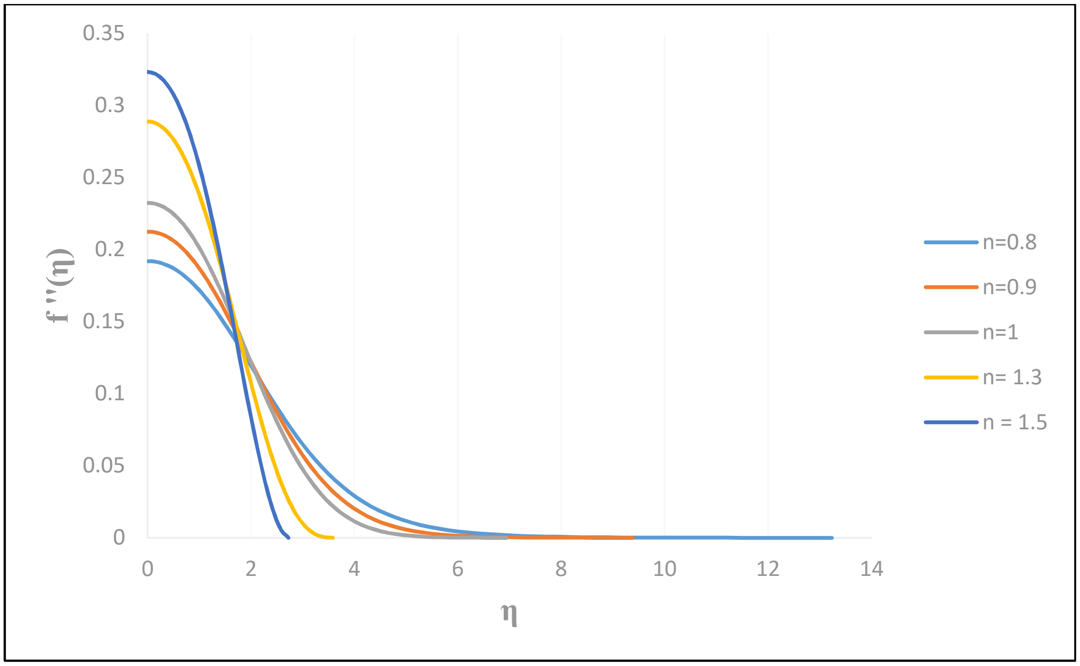

Figure 9. Shear stress profiled for different n values for λ = - 0.5

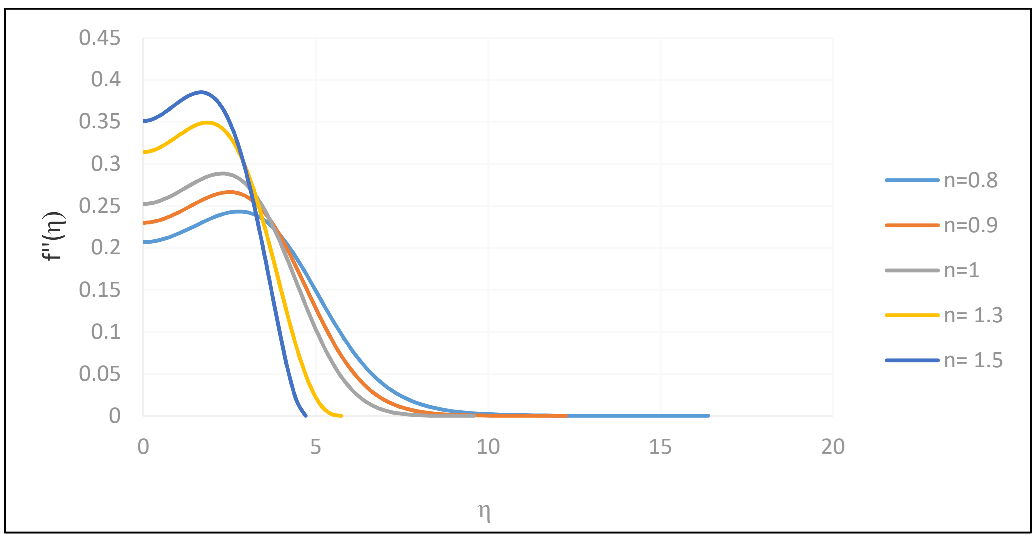

Figure 10. Shear stress profiles for different n values for λ = 0.3

The wall shear stress f ” (0) as a function of n for different values of λ is shown in Figure 11. For comparison, the values obtained by Bognar for λ=0 is also shown. The agreement is satisfactory. The extent of the boundary layer in terms of ‘ η_∞ ‘ , as obtained by the adaptive boundary technique is presented in Figures 12. It is seen that as ‘ n ‘ increases the extent of boundary layer decreases. It is also seen that for Newtonian fluids η_∞ as computed by the present procedure matches well with a typical values of around 6 for Blasius flow. (i.e. $\lambda = 0$ and n=1). It is also seen that these values agree well with the trends predicted by Bognar for $\lambda = 0$, using the iterative transformation method [5], Finally figure 13 shows the wall shear stress as a function λ for Newtonian fluid. It is observed that for $\lambda > 0.35$ ($= \lambda_c$) there is flow separation and hence the wall shear stress disappears. This is the critical value of the wall velocity. Finally the existence of dual solutions is illustrated for Newtonian fluids (n=1) and for $\lambda = 0.15$ is illustrated in Figure 14. As expected the boundary layer thickness is much larger ($\eta_\infty \approx 20$) and the wall shear stress is much smaller (f ”(0) $\approx$ 0.006). These values agree very well with those reported by Bognar in [6,7].

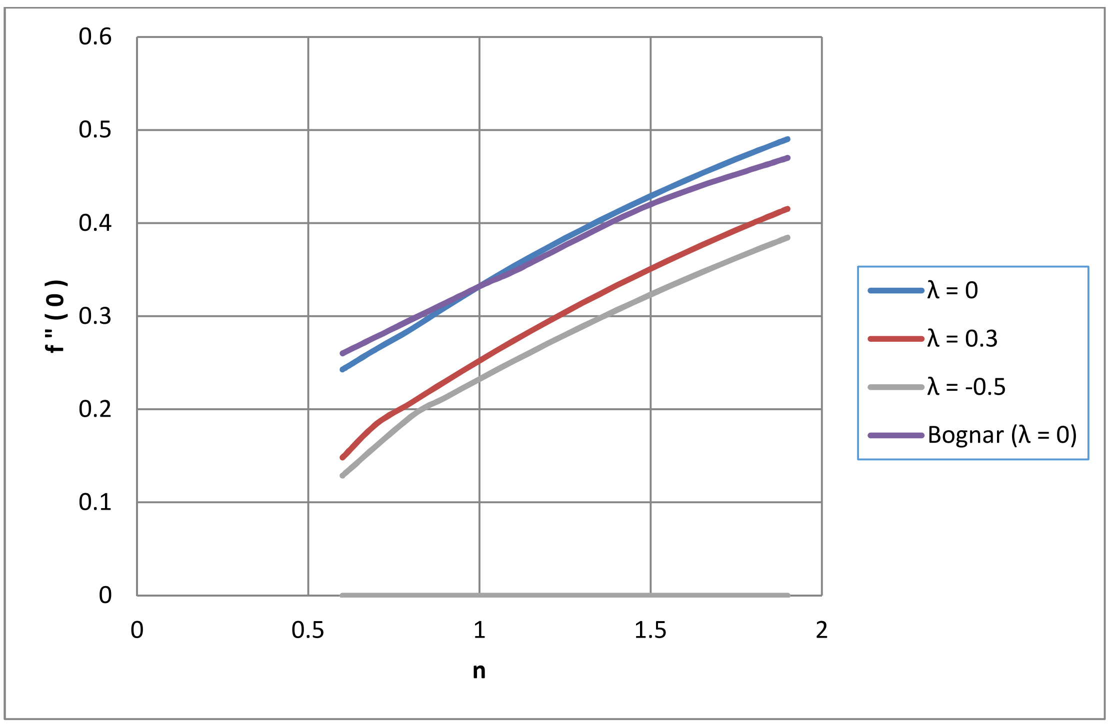

Figure 11. Wall shear stress for different n values

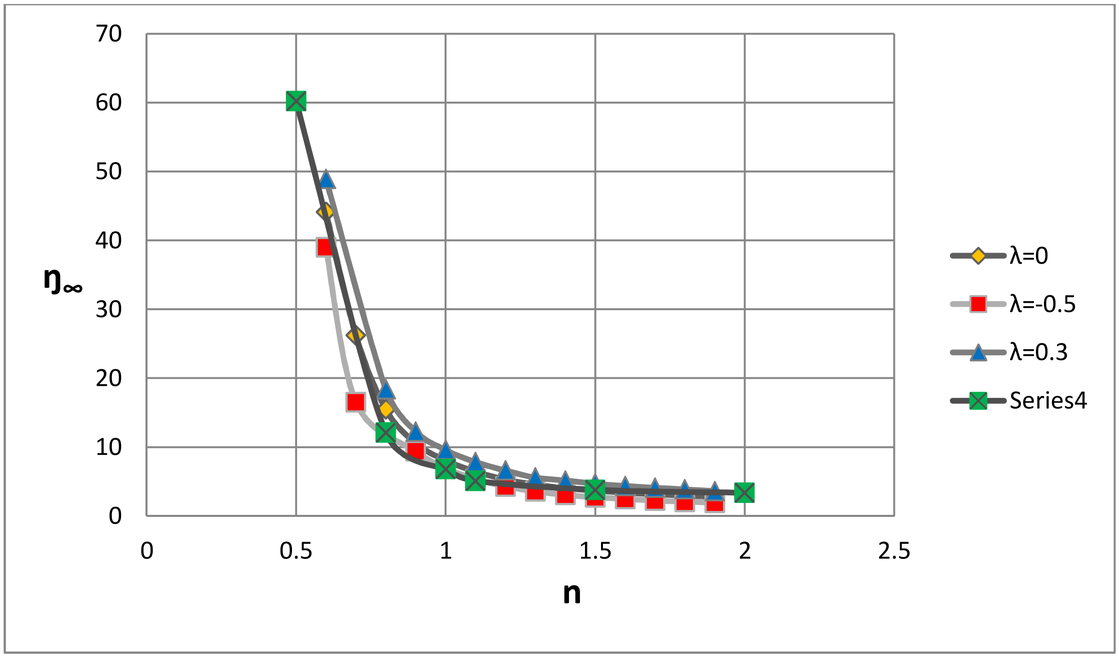

Figure 12. Boundary layer thickness η_∞ as a function of 'n' for different λ

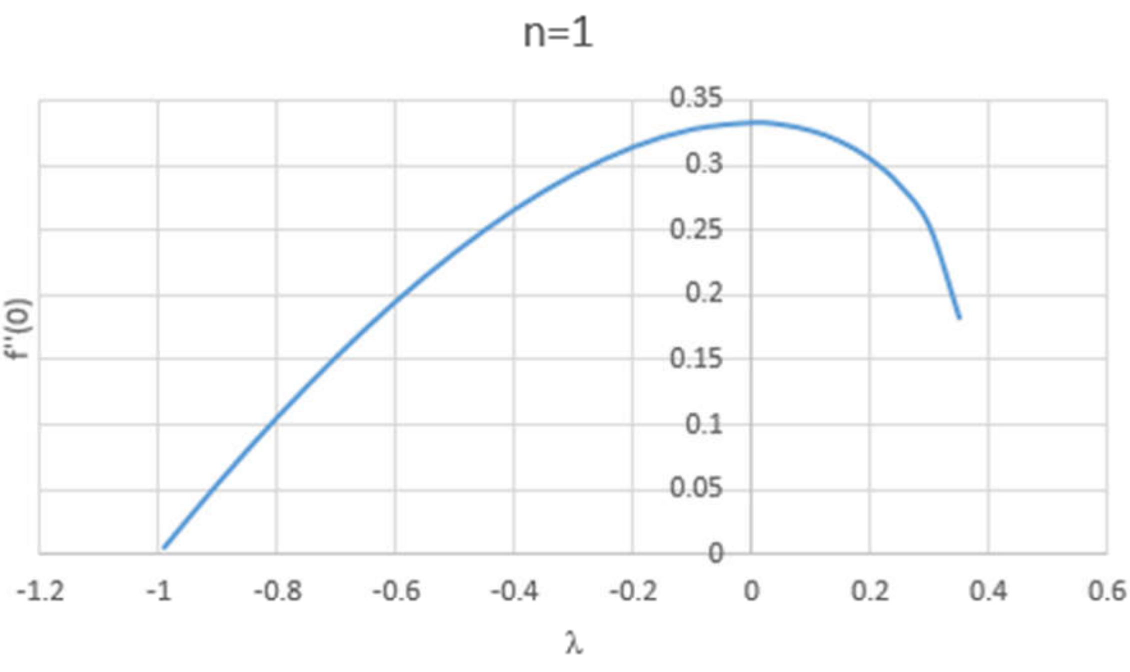

Figure 13. Wall shear stress as a function of λ for Newtonian Fluids

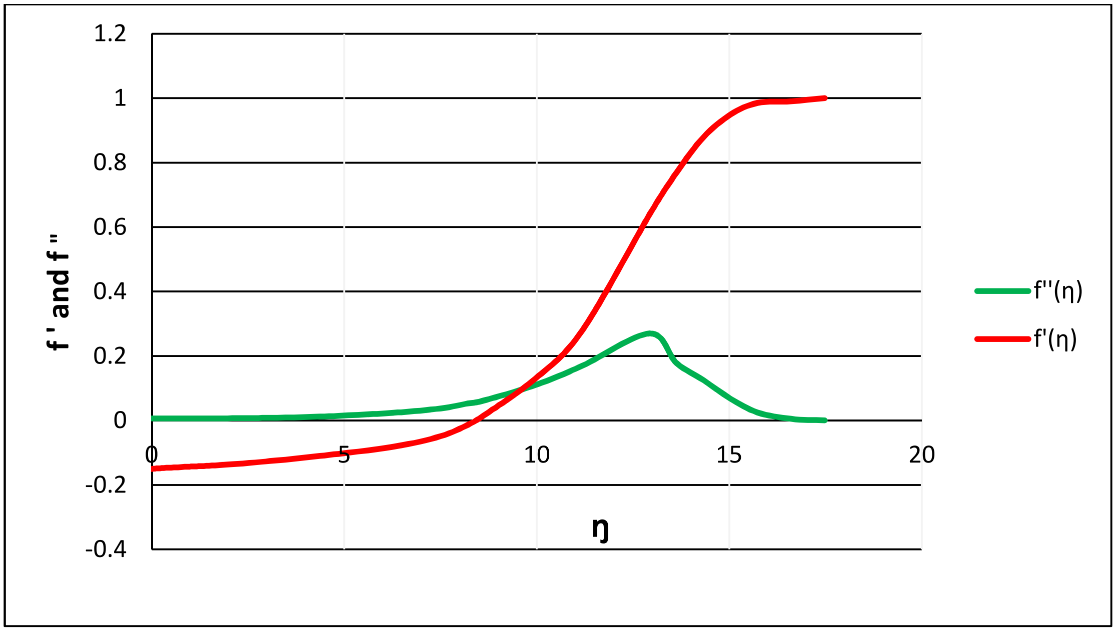

Figure 14. Illustration of dual solution for Newtonian fluids (n=1), λ=0.15

6. Conclusions

An adaptive free boundary technique is used in the present study for solving non-Newtonian fluid boundary layer flows on a moving flat surface. The results obtained in terms of velocity and shear stress profiles for different ' n ' and ' λ ' values, show the correct trends reported in the literature and also the extent of the boundary layer is predicted automatically as part of the solution. This method is quite general and can be applied to solve more complex fluid flow problems that can be formulated as non linear two point boundary value problems in a straightforward manner.

Appendix

Equations (25) and (26) can be put in the general form

$$\frac{dQ}{d\eta} = G(Q,\eta) \quad \eta \in [a,\infty] \qquad (A.1)$$

with the associated boundary conditions given by (27)-(28) also in general form as

$$g[Q(a),Q(\infty)] = 0 \qquad (A.2)$$

Here Q is a n-dimensional vector having ' n ' components

$$Q = Q(q_1, q_2, q_3, q_i, \dots q_{n-1}, q_n) \qquad (A.3)$$

and 'a' and ∞ are the lower and upper boundaries of the semi-infinite region. For the present problem, we replace 'a' by 0 (wall) and ∞ by a finite value η_e. (boundary-layer edge) Equation (A.2) represents general coupled boundary conditions and the uncoupled boundary conditions, as in the present case, can be rewritten as

$$Q_i(0) = C_i \qquad i = 1, 2, \dots\dots r \qquad (A.4)$$

$$Q_{im}(\eta_e) = C_{im} \qquad i_m = 1, 2, \dots\dots (n - r) \qquad (A.5)$$

Here it is assumed that 'r' boundary conditions are prescribed at $\eta = 0$ and the remaining (n-r) boundary conditions are specified at $\eta = \eta_\infty$. For the present problem, $q_1 = f$, $q_2 = u$, $q_3 = v$ and $q_4 = \eta_\infty$. . For the boundary conditions, $r = 2$ and $C_1 = C_2 = 0$ at $\eta = 0$ and at $\eta = \eta_\infty$, $i_1 = 2$, $i_2 = 4$ and and $C_2 = 1$ and $C_4 = \delta$ as specified by equation (28). The non-uniformly space grid points between the wall and the boundary layer edge are generated by specifying the spacing of the first grid point $\Delta\eta_1$ and a multiplying factor ' K ', which is slightly more than unity. Thus the subsequent grid spacings are calculated by

$$\Delta\eta_j = K\, \Delta\eta_{j-1} \qquad j = 1,2,3\dots. J \qquad (A.6)$$

For uniform grid spacing, K = 1 and the grid spacing is given by

$$\Delta\eta = \frac{\eta_e}{(J-1)} \qquad (A.7)$$

where 'J' is the number of grid points. The grid cell centers are defined by

$$\eta_{j+\frac{1}{2}} = {}^{1}\!/_{2}\left(\eta_{j+1} + \eta_{j-1}\right) \qquad (A.8)$$

and the function '$G_{j+\frac{1}{2}}$ ' at this point is defined as

$$G_i^{j+\frac{1}{2}} = {}^{1}\!/_{2}\left(G_i^{j+1} + G_i^{j-1}\right) \qquad (A.9)$$

The differencing of (A.1) is done at these centre points and it is rewritten in finite difference form as

$$Q_i^{j+1} - Q_i^{j-1} = \Delta\eta G_i^{j+\frac{1}{2}}$$

$$= \frac{\Delta\eta}{2}\left(G_i^{j+1} + G_i^{j-1}\right) \qquad (A.10)$$

The above differencing scheme retains second order accuracy even on non-uniformly spaced grids. An iterative process is set up, starting from an initial guess profile for all the n-components of vector Q, as shown below:

$$Q_i^{(j)(q+1)} = Q_i^{(j)(q)} + \delta Q_i^{(j)(q)}$$

$$i = 1,2,3, \dots. n\,;\; j = 1,2,3 \dots J \text{ and } q = 0,1,2\dots \qquad (A.11)$$

In (A.11) superscript (q) refers to the iteration number, (j) refers to the grid point and subscript (i) refers to the i[th] component of the Q-vector. Substituting (A.11) into (A.10) and neglecting higher order terms of $\delta Q_i^{j(q)}$, in the Taylor's series expansion of ' G ' with respect to ' Q ', we get,

$$\delta Q_i^{(j+1)(q)} - \delta Q_i^{(j)(q)} = F_i^{j(q)} \qquad (A.12)$$

where

$$F_i^{j(q)} = -\left(Q_i^{(j+1)(q)} - Q_i^{(j)(q)}\right) + \frac{\Delta\eta}{2}\left(G_i^{(j-1)(q)} + G_i^{(j+1)(q)}\right) + \frac{\Delta\eta}{2}\left(\sum_{i=1}^{n}\frac{\partial G}{\partial Q}\,\delta Q_i^{(j+1)(q)} + \sum_{i=1}^{n}\frac{\partial G}{\partial Q}\,\delta Q_i^{(j-1)(q)}\right) \qquad (A.13)$$

It is to be observed that the incremental vector $\delta Q_i^{j(q)}$ is linear, even though the governing equations are non-linear. Equation (A.12) can be put in the form

$$R_j\,\delta Q^{j+1} - S_j\,\delta Q^j = P_j \qquad (A.14)$$

with

$$P_j = (F_1, F_2, F_3, \dots F_n)_j \qquad (A.15)$$

Equation (A.14) can be further put in block tri-diagonal form

A=LU (A.16)

and this can be solved efficiently by the Thomas algorithm (8). In the present problem, the incremental vector δQ_i^j represents the increments in the variables ($\delta f, \delta u, \delta v, \delta\eta_\infty$). The iterations are carried over till all the components of the incremental vector fall below a prescribed accuracy 'ε ' which typically has a value of $10^{-4.}$ As noted earlier, the method requires guess profiles for stream function (f) , velocity (u) and the shear stress (v), between the wall and the edge of the boundary layer. The velocity is taken as linear and the stream function and shear stress profiles are got by integration and differentiation respectively as

$$f = \frac{\eta^2}{2\eta_\infty}$$

$$f' = \frac{u}{u_\infty} = \frac{\eta}{\eta_\infty}$$

(A.17)

$$f'' = \frac{1}{\eta_\infty}$$

The non-uniformly spaced grid is used to get better resolution near the wall so that the gradients in the velocity and the shear stress are computed accurately. The application of the box scheme at each of the 'J' grid points leads to a large system of algebraic equations, which can be put in block tridiagonal form as

A δQ= B .(A.18)

This can be efficiently solved by LU decomposition method and using Thomas algorithm. In this method, the matrix A is decomposed into a lower triangular L and an upper triangular U matrices. The solution is got by one forward sweep and one backward sweep. This is explained below. Let

$$A = LU \qquad (A.19)$$

where

$$L = \begin{vmatrix} \alpha_1 C_1 \;\; 0\;0\;0\;0 \ldots\ldots\ldots 0\;\;0\;\;0 \\ B_1\alpha_2 \;\; 0\;0\;0\;0\; \ldots\ldots 0\;\;0\;\;0\;\;0 \\ \ldots\ldots\ldots\;\; \ldots\ldots B_j\alpha_j \;\; \ldots\ldots 0\;0\;0\;0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\;\; B_{J-1}\alpha_{J-1} \end{vmatrix} \qquad (A.20)$$

and

$$U = \begin{vmatrix} I \;\;\; \ldots\ldots\ldots\;\; \ldots\ldots\ldots\ldots\ldots \\ 0 \;\;\; I\;\Gamma_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0\;\;0\;\;0\;\; \ldots\ldots I\;\Gamma_j \;\; \ldots\ldots\ldots\ldots\ldots \\ 0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\;0\;0\;0\;0\;0\;0\;I \end{vmatrix} \qquad (A.21)$$

Writing equation (A.18) as $LU\delta Q = B$ and letting $U\delta Q = W$, then it follows that LW=B

The forward sweep (from 1 to J-1) begins with

$$\alpha_1 = C_1$$

$$\alpha_j \Gamma_j = C_j \qquad 1 \le j \le (J-2) \qquad (A.22)$$

$$\alpha_j = A_j - B_j \Gamma_{j-1} \qquad 2 \le j \le (J-1)$$

The backward sweep (from J-2 to 1) continues as

$$\delta Q_{J-1} = W_{J-1}$$

$$\delta Q_j = W_j - \Gamma_{j+1} \qquad (J-2) \ge j \ge 1 \qquad (A.23)$$

This gives the incremental vector δQ_j at all the grid points which is added to the current solution (equation A.11) and iterated till all the components of the incremental vector are less than a prescribed tolerance of the order of 10^{-4} or less.

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