EXISTENCE OF SOLUTIONS OF CAUCHY INITIAL VALUE PROBLEM FOR FUZZY DIFFERENTIAL EQUATIONS

T.Jeyaraj[1] & D.Rajan[2]

[1&2] PG & Research Department of Mathematics, T.B.M.L.College, Porayar-609 307

[1]email : jrajkadali@gmail.com

[2]email : dan_rajan@rediffmail.com

Abstract

In this paper we proved the existence and uniqueness of a solution of the Fuzzy Cauchy's Initial value problem $y'(t) = T\big(t, y(t)\big),\ y(t_0) = y_0$ by using the method of successive approximation..

Keywords: Fuzzy initial value problem (FIVP), levelwise continuous, fuzzy set valued function, Fuzzy solution.

1. Introduction

The fuzzy initial value problem was regularly studied by O. Kelva [6] and by S. Seikkala [10] et. al., The Fuzzy valued functions was developed in recent years by Dubois and Prade [5] and Puri and Ralescu [9]. In this paper we studied the differential equations for fuzzy-valued functions and by generalize certain basic results of calculus, for instance the fundamental theorem of calculus, to fuzzy-valued functions. Finally we prove the existence and uniqueness of a solution to a fuzzy differential equation $y'(t) = T\big(t, y(t)\big),\ y(t_0) = y_0$ provided f satisfies a Lipschitz condition by using the method of successive approximations.

2. Preliminaries

Let the family of all nonempty compact convex subsets of R^n is denoted by $CV(R^n)$, the addition and scalar multiplication in $CV(R^n)$ operations is defined as follows,

Let $A, B \subset R^n$, The distance between A and B is defined by the Hausdorff metric $d(A, B) = max\{sup_{a\in A}\, inf_{b\in B} \|a - b\|, sup_{b\in B}\, inf_{a\in A} \|a - b\|\}$ where $\|.\|$ denotes the usual Euclidean norm in R^n. Then it is clear that $(CV(R^n), d)$ is a metric space and also complete and separable.

Let $C = [\mathrm{l}, \mathrm{m}] \subset$ R be a compact interval and denote

$L^n = \{ w: R^n \rightarrow [0,1] / w \ \ satisfies\ (i) - (iv) below\}$, where

(i) w is normal, i.e., there exists an $x_0 \in R^n$ such that $w(w_0) = 1$,

(ii) w is fuzzy convex,

(iii) w is upper semi continuous,

(iv) $[\mathrm{w}]^{\mathrm{c}} = \overline{\{\mathrm{x} \in \mathrm{R}^n | \mathrm{w(x)} > 0\}}$ is compact.

For $0 < \alpha \leq 1$, denote $[\mathrm{w}]^{\alpha} = \{\mathrm{x} \in \mathrm{R}^n \mid \mathrm{w(x)} \geq \alpha\}$.

Then from (i) – (iv), it follows that the α-level set $[u]^{\alpha} \in \mathrm{CV}(\mathrm{R}^{\mathrm{n}})$, $\forall\, 0 \leq \alpha \leq 1$.

If $g : \mathrm{R}^{\mathrm{n}} \times \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{n}}$ is a function, then, according to Zadeh's extension principle, we can extend *g* to $L^n \times L^n \rightarrow L^n$ by the equation

$g(x,y)(z) = sup_{z=g(u,v)} min\{x(u), y(v)\}$, and $[g(x,y)]^{\alpha} = g([x]^{\alpha}, [y]^{\alpha})$ for all $x, y \in L^n, 0 \leq \alpha \leq 1$ and *g* is continuous.

Note: For any $x, y \in L^n,\ k \in R,\ 0 \leq \alpha \leq 1$

(i) $[x + y]^{\alpha} = [x]^{\alpha} + [y]^{\alpha}$,

$(ii) [kx]^{\alpha} = k[x]^{\alpha}$.

Definition 2.1 Let $D : L^n \times L^n \rightarrow R_+ \cup \{0\}$ defined by

$D(x,y) = sup_{0 \leq \alpha \leq 1} d([x]^{\alpha}, [y]^{\alpha})$, where *d* is the Hausdorff metric in $\mathrm{CV}(\mathrm{R}^{\mathrm{n}})$.

Clearly D is metric in L^n, and o (L^n, D) is a complete metric space.

Definition 2.2. A function $F : C \rightarrow L^n$ is strongly measurable if, for all $\alpha \in [0,1]$, the set-valued mapping $F_{\alpha} : C \rightarrow \mathrm{CV}(\mathrm{R}^{\mathrm{n}})$ defined by $F_{\alpha}(t) = [F(t)]^{\alpha}$ is $(Lebesgue)$ measurable, when $\mathrm{CV}(\mathrm{R}^{\mathrm{n}})$ is endowed with the topology generated by the Hausdorff metric *d*.

Definition 2.3 A function $F : C \rightarrow L^n$ is called level wise continuous at $t_0 \in C$ if the set-valued mapping $F_{\alpha}(t) = [F(t)]^{\alpha}$ is continuous at $t = t_0$ with respect to the Hausdorff metric *d* for all $\alpha \in [0,1]$.

Definition 2.4 A function $F : C \rightarrow L^n$ is called integrable bounded if there exists an integrable function *h* such that $\|\mathrm{x}\| \leq \mathrm{h(t)}$ for all $x \in F_0(t)$.

Definition 2.5 Let $F : C \rightarrow L^n$. The integral of F over C, denoted by $\int_l^m F(t)\,dt$, is defined levelwise by the equation

$$\left(\int_l^m F(t)\,dt\right)^{\propto} = \int_l^m F(t)\,dt$$

$$= \left\{\int_l^m f(t)\,dt \,/\, f : C \rightarrow R^n \textit{is measuablt selection for } F_\alpha\right\}$$

for all $0 < \alpha \leq 1$.

A strongly measurable and integrable bounded mapping $F : C \rightarrow L^n$ is said to be integrable over C if $\int_l^m F(t)\,dt \in L^n$.

3. Differentiability

Definition 3.1 A function $F : C \rightarrow L^n$ is called *differentiable* at $t_0 \in C$ if there exists a $F'(t_0) \in L^n$ such that the limits,

$\lim_{h\rightarrow 0^+} \frac{F(t_0+h)-F(t_0)}{h}$ and $\lim_{h\rightarrow 0^+} \frac{F(t_0)-F(t_0-h)}{h}$ exists and equal to $F'(t_0)$.

Remark 3.1 If a function $F : C \rightarrow L^n$ is differentiable at $t_0 \in C$ then for any $\alpha \in [0,1]$, the set valued function $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at point t_0 with $DF_\alpha(t_0) = [F'(t)]^\alpha$.

If $F : C \rightarrow L^n$ is differentiable at $t_0 \in C$, then we say that $F'(t_0)$ is the *fuzzy derivative* of F(t) at the point t_0.

Theorem 3.1 Let $\mathrm{F} : \mathrm{C} \rightarrow \mathrm{L}^1$ be differentiable. Denote $\mathrm{F}_\alpha(\mathrm{t}) = [\mathrm{f}_\alpha(\mathrm{t}), \mathrm{g}_\alpha(\mathrm{t})]$. Then f_α and g_α are differentiable and $[\mathrm{F}'(\mathrm{t})]^{\propto} = [\mathrm{f'}_\alpha(\mathrm{t}), \mathrm{g'}_\alpha(\mathrm{t})]$.

Proof Now,

$$[F(t+h) - F(t)]^\alpha = [f_\alpha(t+h) - f_\alpha(t), f_\alpha(t+h) - g_\alpha(t)] \text{ and}$$

similarly $[F(t) - F(t-h)]^\alpha = [f_\alpha(t) - f_\alpha(t-h), g_\alpha(t) - f_\alpha(t-h)]$.

Dividing by h and passing to the limits gives the theorem.

Definition 3.2 A function $f : C \times L^n \rightarrow L^n$ is called levelwise continuous at point $(t_0, x_0) \in C \times L^n$ provided, for any fixed $\alpha \in [0,1]$ and arbitrary $\in > 0$, there exists $\delta(\in, \alpha) > 0$ such that $d([f(t,x)]^{\propto}, [f(t_0,x_0)]^{\propto}) < \in$ whenever $|t - t_0| < \delta(\in, \alpha)$ and $d([x]^\alpha, [x_0]^\alpha) < \delta(\in, \alpha)$ for all $t \in C,\ x \in L^n$.

4. Fuzzy differential equations

Let $f: C \times L^n \to L^n$ be continuous, and consider the Cauchy's initial value Problem $y'(t) = f\big(t, y(t)\big),\ y(t_0) = y_0.$ (1)

Example 4.1 Consider the simple problem $y' = -2ty,\ y(0) = 1.$

Here $f(t, y) = -2ty,\ t_0 = 0,\ y_0 = 1,$ and we can apply methods from a standard ordinary differential equations (ODE) course to show that $y(t) = e^{-t^2}$ is the solution.

Assume that $T : P \times L^n \to L^n$ is levelwise continuous, where the interval $P = \{t : |t - t_0| \le \delta \le a\}$.

Consider the fuzzy differential equation (1.1) and denote $C_0 = P \times B(y_0, b)$, where $a > 0, b > 0, y_0 \in L^n,\ B(y_0, b) = \{y \in L^n | D(y, y_0) \le b\}$

Definition 4.1 A mapping $y : P \to L^n$ is a solution to the problem (1) if and only if it is levelwise continuous and satisfies the integral equation

$y(t) = y_0 + \int_{t_0}^{t} f\big(s, y(s)\big) ds,$ for all $t \in P$.

By the method of successive approximation, let us consider the sequence $\{y_n(t)\}$ such that $y_n(t) = y_0 + \int_{t_0}^{t} f\big(s, y_{n-1}(s)\big) ds$, $n = 1, 2, \ldots,$ (2)

where $y(t_0) = y_0,\ t \in P$.

Theorem 4.1 Let $T: C_0 \to L^n$ be a levelwise continuous function and $d([T(t,u)]^\propto, [T(t,v)]^\propto) \le K(d([u]^\alpha, [v]^\alpha) - L)$, for all $(t,u), (t,v) \in C_0$, $K, L > 0$ is a given constant and $\alpha \in [0,1]$. Then there exists a unique solution $y(t)$ of problem (1) defined on the interval $|t - t_0| \le \delta$, where $\delta = min\left\{a, \frac{b}{M}\right\}$ and $M = D(f(t,u), \theta),\ \theta \in L^n$ such that $\theta(t) = 1$ for $t = 0$ and 0 otherwise and for any $(t,u) \in C_0$. Moreover, there exists a fuzzy set-valued mapping $y : P \to L^n$ such that $D(y_n(t), y(t)) \to 0\ on\ |t - t_0| \le \delta\ as\ n \to \infty$.

Proof. Let $t \in P$, from (1.2), it follows that, for $n = 1$,

$$y_1(t) = y_0 + \int_{t_0}^{t} T\big(s, y_0(s)\big)\, ds$$

$\Longrightarrow$ $y(t)$ is levelwise continuous on $|t - t_0| \leq \delta$. And for any $\alpha \in [0,1]$,

$$d([y_1(t)]^\alpha, [y_0]^\alpha) = d\left(\left[\int_{t_0}^{t} T(s, y_0(s))\, ds\right]^\alpha, 0\right) \leq \int_{t_0}^{t} d\left(\left[T(s, y_0(s))\right]^\alpha, 0\right) ds$$

Therefore, by the definition of *D*, and $|t - t_0| \leq \delta$,

$$D(y_1(t), y_0) \leq M|t - t_0| \leq M\delta = b \text{, where}$$

$M = D(T(t,u), \theta)$, $\theta \in L^n$ and for any $(t,u) \in C_0$.

Now, assume that $y_{n-1}(t)$ is levelwise continuous on $|t - t_0| \leq \delta$, and

$$D(y_{n-1}(t), y_0) \leq M|t - t_0| \leq M\delta = b$$

From (2), can deduce that $y_n(t)$ is levelwise continuous on $|t - t_0| \leq \delta$ and

$$D(y_n(t), y_0) \leq M|t - t_0| \leq M\delta = b$$

Thus, the sequence $\{y_n(t)\}$ of levelwise continuous mappings on $|t - t_0| \leq \delta$ and $(t, y_n(t)) \in C_0$, $n = 1,2, \ldots$.

Hence, there exists a fuzzy set-valued mapping $y : P \rightarrow L^n$ such that $D(y_n(t), y(t)) \rightarrow 0$ uniformly on $|t - t_0| \leq \delta$ as $n \rightarrow \infty$.

For $n = 2$,

$$y_2(t) = y_0 + \int_{t_0}^{t} T(s, y_1(s))\, ds$$

Therefore, for any $\alpha \in [0,1]$.

$$d([y_2(t)]^\alpha, [y_1(t)]^\alpha) = d\left(\left[\int_{t_0}^{t} T(s, y_1(s))\, ds\right]^\alpha, \left[\int_{t_0}^{t} T(s, y_0(s))\, ds\right]^\alpha\right)$$

$$\leq \int_{t_0}^{t} d\left(\left[T(s, y_1(s))\right]^\alpha, \left[T(s, y_0(s))\right]^\alpha\right) ds$$

$$d([y_2(t)]^\alpha, [y_1(t)]^\alpha) \leq \int_{t_0}^{t} K\,(d([y_1(t)]^\alpha, [y_0(t)]^\alpha) - L) ds$$

$$\leq \int_{t_0}^{t} K\, d([y_1(t)]^\alpha, [y_0(t)]^\alpha) ds\,,$$

$\Longrightarrow D(y_2(t), y_1(t)) \leq K \int_{t_0}^{t} D(y_1(s), y_0(s))\, ds$

Now, $D\big(y_2(t), y_1(t)\big) \leq KM \frac{|t-t_0|^2}{2!} \leq KM \frac{\delta^2}{2!}$.

In general,

$D(y_n(t), y_{n-1}(t)) \leq K^{n-1}M\frac{|t-t_0|^n}{n!} \leq K^{n-1}M\ \frac{\delta^n}{n!}$.

Indeed, from (2) and for any $\alpha \in [0,1]$, it follows that

$$d([y_{n+1}(t)]^\propto, [y_n(t)]^\alpha) = d\left(\left[\int_{t_0}^{t} T(s, y_n(s))ds\right]^\alpha, \left[\int_{t_0}^{t} T(s, y_{n-1}(s))ds\right]^\alpha\right)$$

$$\leq \int_{t_0}^{t} d\left(\left[T\left(s, y_n(s)\right)\right]^\alpha, \left[T\left(s, y_{n-1}(s)\right)\right]^\alpha\right)ds$$

$$\leq \int_{t_0}^{t} K(d([y_n(s)]^\alpha, [y_{n-1}(s)]^\alpha) - L)\, ds$$

$$\leq K\int_{t_0}^{t} d([y_n(s)]^\alpha, [y_{n-1}(s)]^\alpha)\, ds$$

Thus,

$$D(y_{n+1}(t), y_n(t)) \leq K\int_{t_0}^{t} D(\,y_n(s), y_{n-1}(s))ds$$

$$D\left(y_{n+1}(t), y_n(t)\right) \leq\ MK^n\int_{t_0}^{t}\frac{|s-t_0|^n}{n!}ds$$

$$=\ MK^n\frac{|t-t_0|^{n+1}}{(n+1)!}\ \leq\ MK^n\frac{\delta^{n+1}}{(n+1)!}$$

Consequently, for n = 1,2,...

$$D\left(y_n(t), y_{n-1}(t)\right) \leq \frac{M}{K}\frac{(K\delta)^n}{(n)!}.$$

Let us mention now that

$$y_n(t)\ =\ y_0 + [y_1(t) - y_0] + \cdot\ \cdot\ \cdot + [y_n(t) - y_{n-1}(t)],$$

$\Longrightarrow$ The sequence $\{y_n(t)\}$ and the series

$$x_0 + \sum_{n=1}^{\infty}[y_n(t) - y_{n-1}(t)]$$

have the same convergence properties. By the convergence criterion of Weierstrass, it follows that the series having the general term $y_n(t) - y_{n-1}(t)$, so $D(y_n(t) - y_{n-1}(t)) \to 0$ uniformly on $|t - t_0|\ \leq\ \delta$ as $n\to\infty$.

Hence, there exists a fuzzy set-valued mapping $y : P \to L^n$ such that

$D(y_n(t) - y_{n-1}(t)) \to 0$ uniformly on $|t - t_0| \leq \delta$ as $n \to \infty$.

Now, we get,

$$d\left(\left[T\left(t, y_n(t)\right)\right]^\alpha, \left[T\left(t, y(t)\right)\right]^\alpha\right) \leq K(d([y_n(t)]^\alpha, [y(t))]^\alpha) - L)$$

$$\leq Kd([y_n(t)]^\alpha, [y(t))]^\alpha)$$

for any $\alpha \in [0,1]$., and by the definition of D,

$D\left(T\left(t, y_n(t)\right), T(t, y(t))\right) \leq KD\left(y_n(t), y(t)\right) \to 0$ uniformly on $|t - t_0| \leq \delta$ as $n \to \infty$.

Hence,

$y(t) = y_0 + \int_{t_0}^{t} T\left(s, y(s)\right) ds$ which is the least one levelwise continuous solution of (1).

Uniqueness:

Suppose, $x(t) = y_0 + \int_{t_0}^{t} T\left(s, y(s)\right) ds$ on $|t - t_0| \leq \delta$ another solution of (1), it follows that $D\left(x(t), y(t)\right) = 0$.

Now, for any $\alpha \in [0,1]$,

$$d([y(t)]^\propto, [x_n(t)]^\alpha) = d\left(\left[\int_{t_0}^{t} f(s, y(s)) ds\right]^\alpha, \left[\int_{t_0}^{t} f(s, x_{n-1}(s)) ds\right]^\alpha\right)$$

$$\leq \int_{t_0}^{t} d([f(s, y(s))]^\alpha, [f(s, x_{n-1}(s))]^\alpha) ds$$

$$\leq \int_{t_0}^{t} K(d([y(s)]^\alpha, [x_{n-1}(s)]^\alpha) - L)\, ds$$

$$\leq K \int_{t_0}^{t} d([y(s)]^\alpha, [x_{n-1}(s)]^\alpha)\, ds, \; n = 1,2, \ldots.$$

Hence, $D\left(y(t), x_n(t)\right) \leq K \int_{t_0}^{t} D(y(s), x_{n-1}(s))\, ds$, $n = 1,2,\ldots$

But $D(y(t), x_0) \leq b$ on $|t - t_0| \leq \delta$, y(t) being a solution of (1.3).

It follows that $D\left(y(t), x_1(t)\right) \leq bK|t - t_0|$ on $|t - t_0| \leq \delta$.

Now, assume that,

$$D\big(y(t), x_n(t)\big) \leq \; bK^n \frac{|t-t_0|^n}{n!} \tag{3}$$

on the interval $|t - t_0| \; \leq \; \delta$.

From $D\big(y(t), x_{n+1}(t)\big) \leq \; K \int_{t_0}^{t} D(y(s), x_n(s))\, ds$ and (1.4),

$D(y(t), x_{n+1}(t)) \leq \; bK^{n+1} \frac{|t-t_0|^{n+1}}{(n+1)!}$.

Consequently, for any n, this leads to the conclusion that

$D(y(t), x_n(t)) = \; D(x(t), x_n(t)) \to 0$ on the interval $|t - t_0| \; \leq \; \delta$ as $n \to \infty$.

This proves that, Cauchy's initial value problem (1) has a uniqueness solution.

Corollary 4.1 Let $T: C_0 \to L^n$ be a levelwise continuous function and $d([T(t,u)]^{\propto}, [T(t,v)]^{\propto}) \leq Kd([u]^{\alpha}, [v]^{\alpha}) \;\; \forall \; (t,u), (t,v) \in C_0, \;\; K > 0$ is a given constant and $\alpha \; \in \; [0,1]$. Then there exists a unique solution $y(t)$ of (1.1) defined on the interval $|t - t_0| \leq \; \delta = min\left\{a, \frac{b}{M}\right\}$, $M = D(f(t,u), \theta)$, $\theta \in L^n$ such that θ(t) = 1 for t = 0 and 0 otherwise and for any $(t,u) \; \in C_0$. Moreover, there exists a fuzzy set-valued mapping $y : \; P \; \to L^n$ such that $D(y_n(t), y(t)) \to 0$ on $|t - t_0| \; \leq \; \delta \; as \; n \to \infty$.

Proof The proof of the Corollary is immediate by taking $L = 0$ in the above theorem 4.1.

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